Outline

1. Cellular Automata
   - M/M/1 Queue
   - The asymmetric exclusion process
   - Nagel-Schreckenberg process

2. Two-dimensional models
   - NetNasch
   - Open problems
Collaborators

- Tim Garoni (Monash University)
- **Lele (Joyce) Zhang** (Monash University)
- Somayeh Shiri (Monash University)
- Omar Rojas (La Trobe University)
- Andrea Bedini (University of Melbourne)
- Caley Finn (University of Melbourne)
- John Foxcroft (University of Melbourne)
- ...
- VicRoads
(2008) ARC Centre of Excellence MASCOS begin working with VicRoads
  - Six-month AMSI Industry Internship with VicRoads
History

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  - Stochastic cellular automaton model of traffic in networks
  - Model is discrete in space, time, and state variables
  - Generalizes models studied in Mathematical Physics

- (2012 - 2014) ARC Linkage Project
  - Implemented realistic traffic signals into model (SCATS)
  - Various versions of SCATS, including following features:
    - Adaptive cycle times, splits and offsets
    - Green wave
    - Tram priority processes
    - Pedestrian signals

- (2015 - ) ARC CoE ACEMS/VicRoads develop real world applications
  - NetNaSch, a versatile model for road traffic networks
  - CCellular Automaton Simulator for Arterial Roads (CEASAR)
  - Inner North Road and Tram Network
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M/M/1 Queue

Arrival rate $\alpha$
Service rate $\beta$
M/M/1 Queue

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M/M/1 Queue

Arrival rate $\alpha$

Service rate $\beta$

$\alpha$
M/M/1 Queue

Arrival rate $\alpha$
Service rate $\beta$

Length probability distribution function $P_n$ obeys master equation

\[ \tilde{P}_0 = \beta P_1 + (1 - \alpha)P_0 \]
\[ \tilde{P}_n = \alpha P_{n-1} + \beta P_{n+1} + (1 - \alpha - \beta)P_n \quad n > 0. \]

Factorised stationary solution:

\[ P_n = \left(1 - \frac{\alpha}{\beta}\right) \left(\frac{\alpha}{\beta}\right)^n \quad \alpha < \beta. \]
Asymmetric simple exclusion process (ASEP)

- One-dimensional stochastic cellular automata very popular in statistical mechanics starting in 1990s
- Such models do a reasonable job of explaining qualitative behaviour of freeway traffic
- Spontaneous jams emerge as consequence of collective behaviour
- Advanced analytical methods (integrability, random matrix theory)

If $x_1(t) = 0$, then with probability $\alpha$, $x_1(t+1) = 1$
If $x_L(t) = 1$, then with probability $\beta$, $x_L(t+1) = 0$
If $x_i(t) = 1$ and $x_{i+1}(t) = 0$ then with probability $p$, $x_i(t+1) = 0$ and $x_{i+1}(t) = 1$
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\text{If } x_i(t) = 1 \text{ and } x_{i+1}(t) = 0 \text{ then with probability } p, \ x_i(t + 1) &= 0 \text{ and } x_{i+1}(t) = 1
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\[
\begin{array}{cccccccc}
 & & & & & & & \uparrow \\
\text{p}^2
\end{array}
\]

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Asymmetric simple exclusion process (ASEP)

Stationary state no longer simply factorised
Asymmetric simple exclusion process (ASEP)

Stationary state no longer simply factorised

but can use **matrix** product formalism

\[
P_L(x_1, \ldots, x_L) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} [x_i D + (1 - x_i) E] | V \rangle,
\]

with normalisation

\[
Z_L = \langle W | (D + E)^L | V \rangle = \langle W | C^N | V \rangle, \quad C = D + E
\]

For example,

\[
P_5(10110) = \frac{1}{Z_5} \langle W | DEDDE | V \rangle
\]
If matrices are known, it is easy to compute quantities of interest.

- **Density:**

\[
\rho_i = \mathbb{E}[X_i] = \frac{1}{Z_N} \langle W | C^{i-1} D C^{N-i} | V \rangle
\]
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  \rho_i = \mathbb{E}[x_i] = \frac{1}{Z_N} \langle W | C_i^{-1} D C^{i-1} | V \rangle
  \]

- **Current:**
  \[
  J_{i,i+1} = p \mathbb{E}[\tau_i (1 - \tau_{i+1})] = \frac{p}{Z_N} \langle W | C_i^{-1} D E C^{i-1} | V \rangle
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  \]

The matrices satisfy the relations:

\[
pDE = D + E,
\]

\[
\langle W | E = \frac{1}{\alpha} \langle W |,
\]

\[
D | V \rangle = \frac{1}{\beta} | V \rangle.
\]

Hence

\[
J_{i, i+1} = J = \frac{p}{Z_N} \langle W | C^{i-1} DEC^{N-i-1} | V \rangle = \frac{Z_{N-1}}{Z_N}
\]
Explicit matrix representation

\[ D = \frac{1}{p} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad E = \frac{1}{p} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]
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0 & 0 & 1 & 1 & \\
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\vdots & & & & \\
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0 & 1 & 1 & 0 & \\
0 & 0 & 1 & 0 & \\
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\]

\[
\langle W \rangle = \frac{\kappa}{p} (1, a, a^2, a^3, \ldots), \quad |V\rangle = \frac{\bar{\kappa}}{p} (1, b, b^2, b^3, \ldots)^T
\]

where

\[
a = \frac{1 - \alpha}{\alpha}, \quad b = \frac{1 - \beta}{\beta}, \quad \kappa = \frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\alpha\beta}.
\]
ASEP Phase diagram

Analytic expressions for
- Stationary state
- Density and current profiles
- Relaxation rates
- Fluctuations, large deviations
Mathematical techniques

- Stationary state: Matrix product states
- Relaxation rates: Eigenvalues of transition matrix $\Rightarrow$ Integrability
- Large deviations, current generating functions: Random matrix techniques, Integrability

Example:

Let $Q_1(t)$ be the total time-integrated current, i.e., the net number of particle jumps between the left boundary reservoir and site 1 in the time interval $[0, t]$.

Moments are encoded in $\langle e^{\lambda Q_1(t)} \rangle$ (average over histories)

Theorem (Current fluctuations)

In the low density regime

$$E(\lambda) := \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \langle e^{\lambda Q_1(t)} \rangle = p \frac{e^{\lambda} - 1}{(1 + a)(e^{\lambda} + a)}.$$
Mathematical techniques

- Stationary state: Matrix product states
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Example: **Current large deviation function**

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**Theorem (Current fluctuations)**

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$$E(\lambda) := \lim_{N \to \infty} \lim_{t \to \infty} \log \frac{1}{t} \langle e^{\lambda Q_1(t)} \rangle = p \frac{a(e^\lambda - 1)}{(1 + a)(e^\lambda + a)}.$$
Nagel-Schreckenberg process

- Generalises ASEP
  - Vehicles can have different speeds $0, 1, \ldots, v_{\text{max}}$

  \[
  \begin{array}{ccccccc}
  & & & 2 & & & 0 & 3 & 2 \\
  \end{array}
  \]

- $x_n$ and $v_n$ denote position and speed of the $n$th vehicle
- $d_n$ denotes the gap in front of the $n$th vehicle
- NaSch rules:
  - $v_n \mapsto \min(v_n + 1, v_{\text{max}})$
  - $v_n \mapsto \min(v_n, d_n)$
  - $v_n \mapsto \max(v_n - 1, 0)$ with probability $p$
  - $x_n \mapsto x_n + v_n$
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[  ] [  ] [  ] 1 0 [ 2 ][ 3 ]
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A comparative study of Macroscopic Fundamental Diagrams of arterial road networks governed by adaptive traffic signal systems

Lele Zhang\textsuperscript{a,b}, Timothy M Garoni\textsuperscript{b,*}, Jan de Gier\textsuperscript{c}

\textsuperscript{a}ARC Centre of Excellence for Mathematics and Statistics of Complex Systems, Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia
\textsuperscript{b}School of Mathematical Sciences, Monash University, Clayton, Victoria 3800, Australia
\textsuperscript{c}Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia
Two-dimensional models

Traffic modelling: network of cellular automata

(a) Road network

(b) Intersection

- master node
- slave node
- non-subsystem node
- boundary node
NetNaSch: A mesoscopic model of arterial networks

- **Ingredients:**
  - Dynamics on lanes modelled with (extended) NaSch model
  - Multiple lanes with lane changing
  - Turning decisions (random)
  - Appropriate rules for how vehicles traverse intersections
    - Sydney Coordinated Adaptive Traffic System (SCATS)

- **Features:**
  - Multiple agent types: cars, trams, buses, bicycles, pedestrians, ...
  - Easily applied to arbitrary network structures
  - Essentially any rules for traffic signals can be implemented
  - Relatively modest amount of input data required
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Goals:
- Statistical mechanics of traffic (phase transitions, non-equilibrium behaviour)
- New scenario analysis, generic behaviour
- Policy development
  - Parking restrictions
  - Change signal systems
  - Tram priority
http://ceasar.acems.org.au
The functional relationship between flow and density is the **fundamental diagram**

- Can be computed analytically for ASEP
- What should happen in a network?
- How should one even **define** network flow? (No prescribed direction)
Macroscopic Fundamental Diagrams

- Simplest idea: relate arithmetic means of link density and flow

- If network has link set \( \Lambda \):
  \[
  \rho = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \rho_\lambda, \quad J = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} J_\lambda
  \]

- \( \rho_\lambda \) is density of link \( \lambda \) and \( J_\lambda \) is its flow

Geroliminis & Daganzo 2008
Empirical data from Yokohama

Buisson & Ladier 2009
Empirical data from Toulouse
Time-dependent demand

- Hysteresis in MFD consequence of heterogeneity

Zhang, G & de Gier 2013
Simulated data
Can a simple model explain hysteresis?

\[
\frac{d\rho_1}{dt} = A(\rho_1) - b_1 J(\rho_1) + J(\rho_2)p_2(1 - \rho_1) - J(\rho_1)p_1(1 - \rho_2),
\]

\[
\frac{d\rho_2}{dt} = A(\rho_2) - b_2 J(\rho_2) - J(\rho_1)p_1(1 - \rho_2) + J(\rho_1)p_2(1 - \rho_1)
\]

Let bin 1 be the boundary and bin 2 the interior.
Loading and recovery phases of the two-bin model provide explanation.
Can a simple model explain hysteresis?

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Loading and recovery phases of the two-bin model provide explanation

**Theorem (Law of traffic)**

*It is easier to jam a network than to resolve it.*
Open problems

- Good theoretical models amenable to non-numerical analysis
- NetNaSch: A realistic and flexible model of traffic in arterial networks
- CEASAR: browser based CELlular Automata Simulator for Arterial Roads
- Real time estimators of network behaviour
- How does driver adaptivity affect the shape of MFDs?
- Study on tram priority in Melbourne
- **Big data**: Integrate stopline data (density), GPS data of probe vehicles and predictive modelling